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LETTER TO THE EDITOR

A note on the replica method in normal statistical mechanics

S F Edwards

Cavendish Laboratory, Madingley Road, Cambridge CB3 0HE, UK

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Abstract. The replica method, designed for the statistical mechanics of amorphous materials, involves evaluating the simultaneous averaging of n systems and studying the limit $n \to 0$. It is shown that one can employ the method, for a fixed integral n, to study the phase change of normal systems such as a ferromagnet. The method throws an interesting light on the self consistency of methods of studying such systems.

The replica method (Edwards 1969, Edwards and Anderson 1975, Emery 1975) is a way of averaging the logarithm that occurs in the statistical mechanics of amorphous materials. It replaces $\log Z$ by Z^n, and identitifies $\log Z$ with the coefficient of n as $n \to 0$. But Z^n, for n a positive integer, can be identified with the partition function of n systems (with the same distribution of the frozen variables which characterise the amorphous state). Although one talks as if n is a positive integer, one must be able to take the limit $n \to 0$ and this is contentious (e.g. Bray and Moore 1978).

The present note looks at the case of a normal material, and studies n a positive integer. Then if

$$\exp(-F/kT) = \int \exp(-H/kT) \tag{1}$$

$$\exp(-nF/kT) = \left(\int \exp(-H/kT)\right)^n \tag{2}$$

$$= \int \exp\left(-\sum_a H^{(a)}/kT\right) \tag{3}$$

where $H^{(a)}$ is the Hamiltonian H written in terms of variables in the ath example. Thus for a classical ferromagnet

$$\exp -F/kT = \int \ldots \int \exp\left(\sum_{ij} J_{ij} \boldsymbol{S}_i \cdot \boldsymbol{S}_j/kT\right) \prod \mathrm{d}\boldsymbol{S}_i \tag{4}$$

$$\exp -nF/kT = \int \ldots \int \exp\left(\sum_a \sum_{ij} J_{ij} \boldsymbol{S}_i^{(a)} \cdot \boldsymbol{S}_j^{(a)}/kT\right) \prod_j \prod_b \mathrm{d}\boldsymbol{S}_j^{(b)}. \tag{5}$$

Now consider a phase change which results in the emergence of a non-zero average $\langle \boldsymbol{S}_i \rangle$ to each spin. Without loss of generality one can assume this mean points in the same direction in each of the n replicas. One can therefore try to characterise the condensed

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phase by taking a method of evaluation based on the average $\langle \boldsymbol{S}_i^{(a)} \cdot \boldsymbol{S}_i^{(b)} \rangle$ being non-zero for $(a) \neq (b)$. Such a method would be to rewrite

$$\sum_a \sum_{ij} J_{ij} \boldsymbol{S}_i^{(a)} \boldsymbol{S}_j^{(a)} = \frac{\mu^2}{n} \sum_a \Big(\sum_i \boldsymbol{S}_i^{(a)} \Big)^2 + \Big\{ \sum_a \sum_{ij} J \boldsymbol{S}_i^{(a)} \boldsymbol{S}_j^{(a)} - (\mu^2/n) \sum_a \Big(\sum_i \boldsymbol{S}_i^{(a)} \Big)^2 \Big\} \tag{6}$$

$$= \mathcal{H}_0 + \{\mathcal{H} - \mathcal{H}_0\}. \tag{7}$$

Then

$$\exp(-nF/kT) = \int \ldots \int \exp(-\mathcal{H}/kT) \tag{8}$$

$$\geqslant \int \ldots \int \exp(-\mathcal{H}_0/kT - \langle \mathcal{H} - \mathcal{H}_0 \rangle / kT)$$

$$= \exp(-nF_0/kT - \langle \mathcal{H} - \mathcal{H}_0 \rangle / kT) \tag{9}$$

where

$$\langle \mathcal{H} - \mathcal{H}_0 \rangle = \int (\mathcal{H} - \mathcal{H}_0) \exp[-(\mathcal{H}_0 - nF_0)/kT]. \tag{10}$$

One can now parametrise the local function in terms of a set of fields ϕ_i, as was done by Edwards and Anderson (1978), which in a few lines of straightforward algebra gives

$$\exp(-nF/kT) = \int \prod_i \mathrm{d}\phi_i \exp[-nZ(\phi_i, \mu)] \Big/ \int \prod \mathrm{d}\phi_i \exp[-nZ_0(\phi_i, \mu)] \tag{11}$$

where

$$kTZ(\boldsymbol{\phi}_i, \mu) = \sum_j \ln B(\boldsymbol{\phi}_j) - \sum \tfrac{1}{2} \boldsymbol{\phi}_j^2 - \{ \sum J_{ij} A(\boldsymbol{\phi}_i) A(\boldsymbol{\phi}_j) - \mu^2 A^2(\boldsymbol{\phi}_i) \} \tag{12}$$

where

$$\begin{aligned} A &= [\cosh \mu |\boldsymbol{\phi}| - (\sinh |\boldsymbol{\phi}| \mu)/|\phi|](\boldsymbol{\phi}/|\boldsymbol{\phi}|) \\ B &= (\sinh \mu |\boldsymbol{\phi}|)/\mu |\boldsymbol{\phi}| \end{aligned} \tag{13}$$

and

$$Z_0 = \sum \boldsymbol{\phi}_j^2/2.$$

The functions A and B are familiar in the mean-field theory of ferromagnetism, and indeed this theory is at once recovered if Z is expanded about the mean-field values

$$\partial Z/\partial \phi_i = 0 \qquad \partial Z/\partial \mu = 0 \tag{14}$$

where the mean $\bar{\phi}_i$ is independent of i.

The interesting point is this: equation (11) must be true for all positive values of integral n, and this means that one trial function (6) can only be consistently used if it is coupled with a mean-field approach. This suggests that the simplest evaluation using replica methods is only consistent in a mean-field system, and better methods of evaluation require a better trial function. The identity employed since it is true of any n may suggest new consistent methods of evaluating the partition function.

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